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Determination of the structural efficiency of quadric and cylindrical surfaces when used as reinforced concrete roof structures

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Abstract. Modern architecture of civil engineering structures made of reinforced concrete is demanding solutions to problems related to design and construction of roofs shaped as creative forms which require an adequate combination of arts and physical-mathematical models. In that sense, the quadric and cylindrical surfaces offer an attractive based on their volumetric expression, historicity, and mathematical feasibility. In this work, seven cylindrical and quadric surfaces named elliptic cylinder, parabolic cylinder, elliptic cone, elliptic paraboloid, hyperbolic paraboloid, ellipsoid and hyperboloid of two sheets have been analysed to find out which of them show to be more efficient when used as reinforced concrete roof structures. To do so, the configuration of each structure has been adjusted in terms of similar stiffness, strength parameters and enclosed volume. The structural efficiency was computed using the results of weight/load ratio, available strength, material consumption and relative stiffness. The solution of the model has been achieved using a combination of exact solutions and numerical methods. To compute the model results a Matlab® code was written and validated using the structural software SAP2000®. The best structural efficiency in terms of stiffness, strength, and materials consumption was obtained for roofs configured using surfaces built from parabolas, i.e., elliptic paraboloid, hyperbolic paraboloid, and parabolic cylinder. The results of this work can be used in future studies and applications related to architectural comfort, structural behaviour and material consumption in construction of roofs.

1. Introduction

The roof of a building is essential to protect its users from the sun, rain, and snow giving them a sense of shelter that goes beyond the climatic reality. Its shape, weight and resistance can have an important influence on the safety of the entire building. Roofs of colosseums, churches, theaters, auditoriums, classrooms, and a myriad of architectural solutions can be benefited by constructions with aesthetic forms that defy design and construction techniques [1,2]. Cylindrical and quadric surfaces offer attractive possibilities to shape the roofs in a safe manner. Such surfaces have well known mathematical expressions and can be used in real constructions taking advantage of the larger stiffness arisen from their form [3-6].

Although there is abundant literature related to the mathematics surfaces and design of roofs, there is a lack of literature that combines those two topics for developing a rational structural design [2]. Structural and architectural efficiency of cylindrical and quadric surfaces when used as reinforced concrete roofs are not well known. In fact, there is no a guide to define the efficiency in that sense. This work is a contribution to overcome that lack.



This research studies the efficiency of cylindrical and quadric surfaces when used as reinforced concrete roof structures. Seven cylindrical and quadric surfaces named elliptic cylinder, parabolic cylinder, elliptic cone, elliptic paraboloid, hyperbolic paraboloid, ellipsoid and hyperboloid of two sheets have been analysed. To simulate a real concrete roof each surface has been considered as an extended body with variable thickness. Loads, stiffness parameters, and boundary values have been applied to a model and results obtained have been used to compute the structural efficiency. Geometrical parameters have been chosen trying to obtain similar enclosed volume contained between the surface and an inferior horizontal plane.

The structural efficiency has been defined in terms of four relations: weight/load ratio, available strength, material consumption and relative stiffness. Such relations have been expressed as relative values with respect to a reference surface. The reference surface has been chosen as that having the most efficient configuration to be used as a roof. Final comparisons have been presented as percentages which describe the proximity of computed values to the reference values. The solution of the model has been achieved using a combination of exact solutions and numerical methods. To validate the model a Matlab® code was written and validated using the structural software SAP2000®.

2. Quadric and cylindrical surfaces used as roofs

In a general form, quadric and cylindrical surfaces can be expressed as show the Equation (1).

$$\alpha \frac{(x-h)^2}{a^2} + \beta \frac{(y-k)^2}{b^2} + \gamma \frac{(z-l)^\epsilon}{c^\epsilon} = \delta, \quad (1)$$

where x, y are the values measured along the horizontal axes; z the values measured along the vertical axis; h, k, l the coordinates of the center or vertex of the curves which shape the surface; α, β, γ coefficients that can take values of 0, 1 or -1; δ is a number that takes values of 0 or 1; ϵ is an exponent that takes values of 1 or 2; and, a, b, c the parametric axes of the curves that shape the surface. Equation (1) describes the locus of any quadric or cylindrical surface used in this work. Cylindrical surfaces are obtained when $\alpha = 0$ and quadric surfaces when $\alpha = 1$ or $\alpha = -1$. Selecting appropriate values for each number, the canonical expression for each one of the surfaces can be written [7,8]. Equation (2) states the locus of an elliptic cylinder extending in the direction of x -axis. It is obtained doing $\alpha = 0, \beta = 1, \gamma = 1, \epsilon = 2, \delta = 1$ in Equation (1).

$$\frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1. \quad (2)$$

Equation (3) states the locus of a parabolic cylinder extending in the direction of x -axis. It is obtained doing $\alpha = 0, \beta = 1, \gamma = 1, \epsilon = 1, \delta = 0$ in Equation (1):

$$\frac{(y-k)^2}{b^2} + \frac{(z-l)}{c} = 0. \quad (3)$$

Equation (4) states the locus of an elliptic cone extending in the direction of z -axis. It is obtained doing $\alpha = 1, \beta = 1, \gamma = -1, \epsilon = 2, \delta = 0$ in Equation (1).

$$\frac{(y-k)^2}{b^2} + \frac{(z-l)}{c} = 0. \quad (4)$$

Equation (5) states the locus of an elliptic paraboloid extending in the direction of z -axis. It is obtained doing $\alpha = 1, \beta = 1, \gamma = 1, \epsilon = 1, \delta = 0$ in Equation (1).

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)}{c} = 0. \quad (5)$$

Equation (6) states the locus of a hyperbolic paraboloid extending in the direction of x-axis. It is obtained doing $\alpha = 1, \beta = -1, \gamma = -1, \epsilon = 1, \delta = 0$ in Equation (1).

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} - \frac{(z-l)}{c} = 0. \quad (6)$$

Equation (7) states the locus of an ellipsoid. Direction of extension depends upon the values of a, b, c. It is obtained doing $\alpha = 1, \beta = 1, \gamma = 1, \epsilon = 2, \delta = 1$ in Equation (1).

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1. \quad (7)$$

Equation (8) states the locus of a hyperboloid of two sheets extending in the direction of z-axis. It is obtained doing $\alpha = -1, \beta = -1, \gamma = 1, \epsilon = 2, \delta = 1$ in Equation (1).

$$-\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1. \quad (8)$$

3. The model of roofs shaped as cylindrical and quadric

A real roof must have adequate strength and stiffness. These properties are generated from the material characteristics, geometrical configuration and volume of steel and concrete [9]. Taking advantage of the stiffness given by cylindrical and quadric surfaces, the thickness can vary from a larger value near the supports to a lower one in the highest center of the surface. To include this condition in the model, one upper and one lower surface have been considered using the same canonical equation but varying its parameters. To enclose a typical volume, each roof was delimited by the plain $z = 0$ as the lower border and by the surface as the upper border. The ordinates (z) of the upper border were obtained doing. Equation (9) allows to compute the surface height at any point (x, y). The total weight of each roof was computed using multiple integral as follows in the Equation (10).

$$z = l + c \left[\frac{1}{\gamma} \left(\delta - \alpha \frac{(x-h)^2}{a^2} - \beta \frac{(y-k)^2}{b^2} \right) \right]^{\frac{1}{\epsilon}}, \quad (9)$$

$$W = \mu \int_{x_f}^x \int_{y_f}^{y_f} \int_{z_i}^{z_f} dz dy dx, \quad (10)$$

where z_i and z_f are the lower and upper surfaces defined by the Equation (9) which delimitate the thickness variation; y_i and y_f are the initial and final border curves which delimitate the intersection of a horizontal plane with the surface; and, x_i and x_f are the initial and final border values corresponding to the domain of x . The infinitesimal volume is generated in Equation (10) as $dz dy dx$. The structural design of a typical slab requires that the steel ratio supplied for attending the bending demand to be low. This condition imposes larger thickness requirements than those needed for attending the shear demands. For this reason, in this work, the verification of the structural safety for each part of the roof has only considered the bending effects. Equation (11) states the safety condition that must be complied when bending moment effect is considered. Then, to satisfy the safety conditions, it must be complied that [10].

$$M_{ava} \geq M_{dem}. \quad (11)$$

Being M_{ava} the available bending strength of a section which depends mainly on the steel ratio and the inner lever arm associated to the concrete thickness, and M_{dem} the demanding bending moment that depends upon the external loads and stiffness configuration of the structure [11,12].

To analyse the roofs, a Matlab® code based on the Direct Stiffness Method was written. Each surface was divided in 576 shells. Values of 1.0 and 1.0 KN/m² were considered for the live load and superimposed dead load, respectively [13]. The own weight was internally computed by the code. Internal forces, deflections, external reactions, and geometrical quantities were obtained as results of the code run. To validate the results, the based on the finite element method software SAP2000® was used.

4. Results and discussion

To facilitate the solution of the model, center and vortex coordinates (h, k, l) were taken as (0,0,0) for each surface. In every case the larger semi-axis *a* direction was alined parallel to the *x* axis. The main section of each roof was obtained cutting it with the plane *y* = 0 resulting to be a vertical curve. Enclosed volume (E.V.), surface area (S), typical length of the main section (L), and mean height (Z) of each roof were computed as basic measurements to make comparisons of their geometrical (architectural) efficiency. The enclosed volume was delimited framing each surface-to-*z* = 0 plane intersection using a 24mx24m quadrate. The elyptic cylinder resulted to have the upper values of basic measurements. Then, to compare the relative behaviour of each structure, results were expressed as a ratio with respect to those obtained for the elliptic cylinder (see Table 1). Here, the resulting numbers are named “relative geometrical measurements (RGM)”.

The Table 1 show the relative geometrical properties (RGM) of studied curves. There are shown only mean values of RGM. Values are written as percentages of relative efficiency compared with that observed for the elliptic cylinder. Coefficients of variation (C.V.) range from 2 to 3% for most surfaces. In the case of elliptic cone, C.V. range from 4 to 6%.

Table 1. Relative geometrical properties (RGM) of studied curves.

Surface Type	L	S	E.V.	Z
Elliptic cylinder	100	100	100	100
Parabolic cylinder	97	97	86	86
Elliptic cone	95	92	62	62
Elliptic paraboloid	97	96	84	84
Hyperbolic paraboloid	97	98	89	89
Ellipsoid	97	98	89	89
Hyperboloid (two sheets)	96	95	81	81

From Table 1, it is evident that the degree of the measurement of each locus imposes its influence upon the relative geometrical measurements. The degree of the measurement of L, S and E.V. is 1,2 and 3 respectively. The average value of each one of the first three columns of Table 1 is 97%, 95% and 82% for L, S and E.V. respectively. Then, the larger the degree of the measurement the lower the relative geometrical property value. Also, it can be seen that the surfaces associated to parabolic forms show the higher numbers and offer architectural advantages related to larger availability of free space. For example, parabolic cylinder, elliptic paraboloid and hyperbolic paraboloid have average RGM values of 97%, 97% and 86% for L, S and E.V. respectively. In contrast, other surfaces have 96%, 95% and 77% for the same parameters.

In practical terms, this information can be related to the architectural comfort and material consumption. For example, the enclosed volume and the surface can be used to make computations related to the control of the air temperature and architectural illumination. In the case of structural applications, measurements related to L and S can be useful when evaluating the weight of the roof.

In this study, the efficiency related to the structural design is associated to adequate strength and stiffness of the roof. In fact, the optimization of the material consumption and construction technics are related to those two characteristics [14,15]. For these reasons, the strength efficiency has been translated to the material consumption needs following the ultimate strength method. The stiffness efficiency was evaluated considering the relative maximum deflection of each roof (L/def.) [13]. This number was calculated dividing the horizontal extension of the main section (Lh) by the computed value of maximum

deflection (def.). For every case $L_h = 24$ m. To impose similar stiffness conditions to each roof, the thickness shell was assigned using four values which were continuously enlarging from the lowest zone to the higher part: 30, 24, 18 and 12 cm. The Table 2 show the results for structural stiffness efficiency.

All cases have been computed using $L_h = 24$ m. Names of surface type have been arranged showing the stiffer surfaces in the upper part of the table.

Table 2 shows numbers related to the structural stiffness efficiency. In general, the larger $L/def.$ value the larger the stiffness roof, *i.e.*, lower $L/def.$ values are associated to surfaces susceptible to larger deformation. Values of $L/def.$ for the elliptic paraboloid and hyperbolic paraboloid are about between 24 and 42 times those obtained for the elliptic cones and elliptic cylinder. This means that the first ones are much stiffer than last ones. In general, the stiffer roofs resulted to be the surfaces associated to parabola (elliptic paraboloid, hyperbolic paraboloid, and parabolic cylinder).

Table 2. Results for structural stiffness efficiency.

Surface Type	Load	def. (mm)	$L_h/def.$
Elliptic paraboloid	Live	0.2	120,000
Hyperbolic paraboloid	Live	0.2	120,000
Parabolic cylinder	Live	0.3	80,000
Hyperboloid (two sheets)	Live	1.0	24,000
Ellipsoid	Live	1.1	21,818
Elliptic cone	Live	4.7	5,106
Elliptic cylinder	Live	8.7	2,759

Figure 1 shows the configuration of modeled surfaces and the deformed surface obtained after the results of validation of the Matlab® code using the program SAP2000®. Non-deformed surface (related to a non-loaded structure) resulted to be a concave function in all cases. In general, two types of deformed shape were identified. The first case showed typical deformed curves M-shaped with a single lower vertex at the center of span suggesting a softer response against vertical displacements in the upper zone. The second case showed typical deformed curves W-shaped with two lower vertices located at each side of the center of span suggesting a softer response against vertical displacements in the lateral zones. Only elliptic cylinder and ellipsoid developed M-shaped deformed curves.

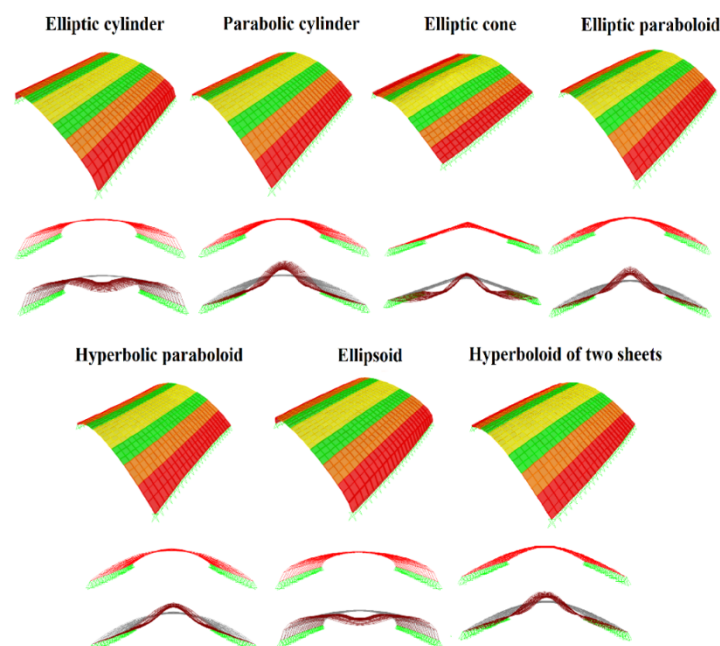


Figure 1. Roofs configuration and deformed surfaces under loads obtained using Sap2000®.

Table 3 show the results for relative material consumption. Needed reinforcement steel volume computed considering standards requirements ($A_s > A_{smin}$) and theoretical requirements (A_s) have been obtained. Also, concrete volume has been reported. In each case, relative volume (Rel. Vol.) is expressed as percentage of that obtained for the elliptic cylinder. C.V. is the coefficient of variation. Table 3 presents the relative material consumption for each roof. Material consumption of reinforcement steel and concrete are closely related to strength efficiency. In general, the lower the relative material consumption, the larger the efficiency. From the theoretical results obtained after solving the model, it can be concluded that the best efficiency is linked to quadric surfaces associated to parabola (hyperbolic paraboloid and elliptic paraboloid) which require only around 35 to 37% of steel (A_s) and 78 to 93 % of concrete of that of the reference surface. On the contrary, elliptic cone has the lowest efficiency demanding 178% of steel (A_s) and 91% of concrete of that of the reference surface.

This finding has been verified considering making a cost analysis in which the cost per volume of steel has varied from 70 to 90 times that of concrete. If, in addition, structural weight is included in evaluation of structural efficiency, it is evident that the roof shaped as hyperbolic paraboloid having the lower concrete volume will be the lighter structure. If the requirements stated for minimum values of steel reinforcement ratio are considered ($A_s > A_{smin}$), previous conclusions are still valid [13].

Table 3. Results for relative material consumption.

Surface Type	Steel ($A_s > A_{smin}$)			Steel (A_s)			Concrete		
	Vol. (m ³)	Rel. Vol.	C.V.	Vol. (m ³)	Rel. Vol.	C.V.	Vol. (m ³)	Rel. Vol.	C.V.
Elliptic cylinder	1.16	100%	22%	0.52	100%	87%	137	100%	0%
Parabolic cylinder	1.18	101%	22%	0.18	35%	83%	129	95%	0%
Elliptic cone	1.40	121%	36%	0.93	178%	85%	124	91%	2%
Elliptic paraboloid	1.17	101%	24%	0.19	37%	84%	127	93%	1%
Hyperbolic paraboloid	1.16	100%	25%	0.18	35%	91%	104	76%	0%
Ellipsoid	1.16	99%	25%	0.49	94%	88%	135	99%	2%
Hyperboloid two sheets	1.15	99%	24%	0.32	62%	95%	127	93%	1%

5. Conclusions

Seven cylindrical and quadric surfaces named elliptic cylinder, parabolic cylinder, elliptic cone, elliptic paraboloid, hyperbolic paraboloid, ellipsoid and hyperboloid of two sheets have been analysed to investigate how efficient are when used as roofs made of reinforced concrete. The results of this work can be useful when studying the architectural comfort, structural behaviour and material consumption in construction of roofs. For example, the enclosed volume and the surface can be used to make computations related to the air temperature control and calculations associated to the architectural illumination. In the case of structural applications, both measurements can be related to the weight of the structure.

For architectural applications, roofs built following the form of an elliptic cylinder offer the largest geometrical parameters (enclosed volume, surface area and average height). Roofs conformed using surfaces associated to parabola (elliptic paraboloid and hyperbolic paraboloid) resulted to have the best structural efficiency in terms of stiffness, strength, and materials consumption. On the contrary, elliptic cone resulted to have the lowest efficiency.

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