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Physical – mathematical model to determine the structural efficiency of sections shaped as ellipses and parallelograms when used in reinforced concrete columns

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Abstract. Current practice of structural design of reinforced concrete columns uses mainly rectangular and circular sections. Among other reasons, this preference is linked to its ease of construction and spatial efficiency, and the historical inertia of civil works processes. Nowadays, modern design and construction techniques make possible to study the structural beahviour of sections distinctively shaped. If typical sections are extended from circular to elliptical and from rectangular to parallelogram shaped, it is possible to compute mathematical and physical parameters needed for the structural analysis. This work presents a physical–mathematical model to determine the structural efficiency in terms of relative strength and stiffness of sections shaped as ellipses and parallelograms when used in reinforced concrete columns. To solve the model, a Matlab® code for drawing the column interaction diagram of each studied section was written. Each diagram describes, in terms of mechanical physics, the relation between the uniaxial bending and the compression load that defines safety conditions as a mathematical limiting curve. The governing equation has the form P(m,g)=f(M(m,g) in which P describes the axial load strength function, M the bending moment strength function, m the material properties, and g the geometrical characteristics function. Structural efficiency has been expressed as the relative behaviour of each section with respect to comparable rectangular sections. To obtain the results of the model, a Matlab® code was written.

1. Introduction

Worldwide, reinforced concrete columns are the vertical load-bearing most used in buildings. Rectangular and circular sections are the preferred sections used in the design and construction of reinforced columns mainly because they are easer to build [1-4]. However, there are other sections shapes which behaviour has not been sufficiently studied yet. The use of elliptical and parallelogram shaped sections in reinforced concrete columns can offer interesting aesthetic, structural and constructive features that broaden the available solutions [5-7].

The circle can be considered as an ellipse which semi-axes take the same value, i.e $a = r$ and $b = r$ being r the circle radius [8,9]. On the other hand, the rectangle can be considered as a parallelogram with four right angles. These mathematical characteristics allow to think in the possibility of using elliptical and parallelogram shaped sections in columns subjected to a combination of uniaxial moment and axial load. The enlarged in one direction nature of these forms confer on themselves a larger moment of inertia which can give a good structural efficiency when the applied bending moment is about the minor axis (perpendicular to the enlarged direction) [10-12].

This work researches the efficiency of elliptical and parallelogram shapes sections when used as sections of reinforced concrete columns. To do so, four sections forms have been studied: circle, ellipse, rectangle, and parallelogram. Variable load and bending moment pairs have been applied to different section geometrical configurations containing variable steel reinforcement ratios. The structural efficiency has been described based on geometrical and structural considerations. Geometrical considerations included the comparison of available area and moment of inertia of each section. Structural considerations were related to available strength and material consumption associated to reinforcement steel ratio. To solve the model a Matlab® code was written.

2. Description of the physical-mathematical model

Rectangular sections were used as reference sections. Vertexes of each rectangular section were located at $(0,-b_r)$, $(0,b_r)$, $(2a_r,b_r)$, and $(2a_r,-b_r)$. Elliptical sections were built using the Equation (1) [8,9], where x,y are the values measured along the horizontal and vertical axis respectively; a semi-major axis, b the semi-minor axis, and $a=\frac{4}{3}b$. Major axis was taken as parallel to y axis (vertical).

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1. \tag{1}$$

Values of a and b were computed using the same section area assigned to rectangular sections. Ordinate value at each point can be determined doing what is written in Equation (2).

$$y=\pm\frac{2b}{a}(x^2+2ax)^{\frac{1}{2}}. \tag{2}$$

Circular sections were built doing a=r and b=r in Equation (1). Parallelogram sections were built using the relationships described in Equation (3).

$$\begin{gathered} y=\pm\frac{b_p}{a_p}x \text{ if } 0\le x\le a_p, \\ y=\mp\frac{b_p}{a_p}x\pm 2b_p \text{ if } a_p<x\le 2a_p. \end{gathered} \tag{3}$$

Vertexes of each parallelogram shaped section were located at (0,0), $(a_p,-b_p)$, (a_p,b_p), and $(2a_p,0)$. Values of a_p and b_p were computed using the same section area assigned to rectangular sections and doing $a_p=\frac{4}{3}b_p$.

The first moment of area of each section was used to determine the position of the inner compression force produced by the concrete C when the column section is subjected to an external force P. First moment of area for each section was computed using Equation (4) [8,9].

$$Q=\int_{x_0}^{x_f} xy_n dx, \tag{4}$$

where x_0 and x_f describe the position of the lower and upper limit of the compressed concrete zone of each section, and y_n is the total height producing a differential area element for a given x value. Then, $x_f=2a$ for elliptical and circular sections, $x_f=2a_p$ for parallelogram shaped sections, and $x_f=2a_r$ for rectangular sections. Area of compressed zone was computed as stated in Equation (5).

$$A_c=\int_{x_0}^{x_f} y dx. \tag{5}$$

Position of force P is given by Equation (6).

$$x_m = \frac{Q}{A_c}. \tag{6}$$

From the static equilibrium of forces, the total reactive force produced by the reinforced concrete section was computed as described by Equation (7).

$$P = \emptyset(0.85f'_c(A_c - A_{sc}) + \sum f_{si} A_{si}), \tag{7}$$

where $\emptyset$ is a variable strength reduction factor that depends on the failure quality [13,14], f'_c is the concrete compressive strength, A_{sc} is the steel reinforcement area located within the compressed area, f_{si} is the stress of the reinforcement bar i, and A_{si} is the reinforcement area of bar i. f_{si} value and sign were determined considering a bilinear stress-strain diagram composed of one elastic zona and one plastic zone [3,4].

The elastic zone shows a proportional stress to strain ratio from nule values to a maximum value associated to the yielding condition. In the plastic zone the yielding stress is sustained until failure. Sign convention was set considering upwards positive forces produced by compression response and counterclockwise positive moments with respect to the plastic centoid of the section. In the same manner, the reactive bending moment produced by the reinforced concrete section was calculated using Equation (8).

$$M = \emptyset(0.85f'_c(A_c - A_{sc})x_m + \sum x_i f_{si} A_{si}), \tag{8}$$

where x_i is the distance from the steel rebar i to the plastic centroid of the section, positive to the right of the plastic centroid. The second moment of area of each section was used to calculate the moment of inertia with respect to y axis (minor moment of inertia) using Equation (9).

$$I_{bb} = \int x^2 y_n dx. \tag{9}$$

In a similar manner, the moment of inertia with respect to x axis (major moment of inertia) can be obtained using Equation (10).

$$I_{aa} = \int y^2 x_n dy, \tag{10}$$

where x_n is the total width producing a differential area element for a given y value.

3. Experimental setup

Structural behaviour a total of 64 reinforced concrete sections was studied. Circular, elliptical, parallelogram shaped, and rectangular sections were investigated using 4 different sizes in each case. At the same time, 4 different steel ratios were applied to each section. To make comparable the results, four identical section area values were used. A wide range of area values was used attempting to include the most practical situations observed in real applications. Small sections sized as lower as 30 cm-side and large sections sized up to 174 cm-side were chosen to apply the model.

Table 1 shows the geometrical characteristics of each section. There, r is the circle radius and A is the section area. The name has been chosen to describe the type of section and its larger section dimension expressed in milimeters.

Figure 1 shows the section of the samples named circle450, ellipse520, parallelogram1302, and rectangle920. Each one of them has a section area of 0.636 m^2. Reinforcement bars arrangements were chosen guaranteeing the same separation between bars within the same section. Steel reinforcement ratio was computed as $p = A_s/A$ where A_s is the total steel area. To give uniformity in the stress distribution, 36 reinforcement bars per section were used in all the cases. Steel reinforcement ratios of 1%, 2%, 3% and 4% were considered complying with current standards [1,2,13,14].

Table 1. Geometrical properties of studied sections.

Number	Name	r(m)	a(m)	b(m)	A(m^2)	Iaa(m^4)	Ibb(m^4)
1	Circle150	0.150			0.071	0.0004	
2	Circle300	0.300			0.283	0.0064	
3	Circle450	0.450			0.636	0.0322	
4	Circle600	0.600			1.131	0.1018	
5	Ellipse173		0.173	0.130	0.071	0.0005	0.0003
6	Ellipse346		0.346	0.260	0.283	0.0085	0.0048
7	Ellipse520		0.520	0.390	0.636	0.0429	0.0242
8	Ellipse693		0.693	0.520	1.131	0.1357	0.0763
9	Parallelogram434		0.217	0.163	0.071	0.0006	0.0003
10	Parallelogram868		0.434	0.326	0.283	0.0089	0.0050
11	Parallelogram1302		0.651	0.488	0.636	0.0450	0.0253
12	Parallelogram1736		0.868	0.651	1.131	0.1421	0.0799
13	Rectangle306		0.153	0.115	0.071	0.0006	0.0003
14	Rectangle614		0.307	0.230	0.283	0.0089	0.0050
15	Rectangle920		0.460	0.345	0.636	0.0450	0.0253
16	Rectangle1228		0.614	0.460	1.131	0.1421	0.0799

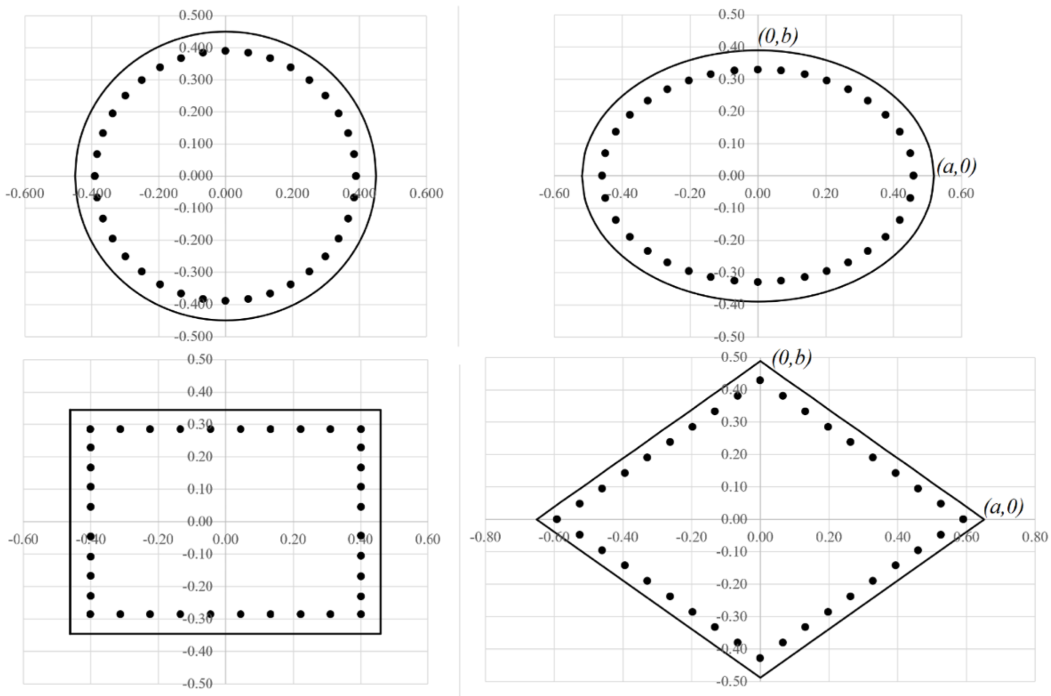

Figure 1. Examples of studied reinforced concrete sections having similar area.

4. Results

To compare the relative behaviour of each section, results were expressed as a ratio with respect to those obtained for the rectangular section. To investigate the structural capacity of each section, Matlab® code for drawing the column interaction diagram was written. Each diagram is a useful tool to understand the relation between the uniaxial bending and the compression load that defines safety conditions as a mathematical limiting curve [15-20]. Structural efficiency was calculated comparing the resisting axial load and bending moment of each section to those values proper of rectangular sections. A Matlab® code was written for assessing such relative strength values.

Figure 2 shows examples of column interaction diagrams obtained for the smallest and largest sections. From parts a and b, it is evident that the maximum bending moment capacity given by a rectangular section can be about 30% larger than that given by a circular one. The same gain is observed in parts Figure 2(c) and Figure 2(d). Furthermore, the nearness of behaviour shown by small rectangular, parallelogram-shaped and elliptic sections (Figure 2(c) and Figure 2(d)) suggests that, in any case, the choice of a circular section could show a lower strength. In general, the maximum bending strength is related to a low axial load capacity.

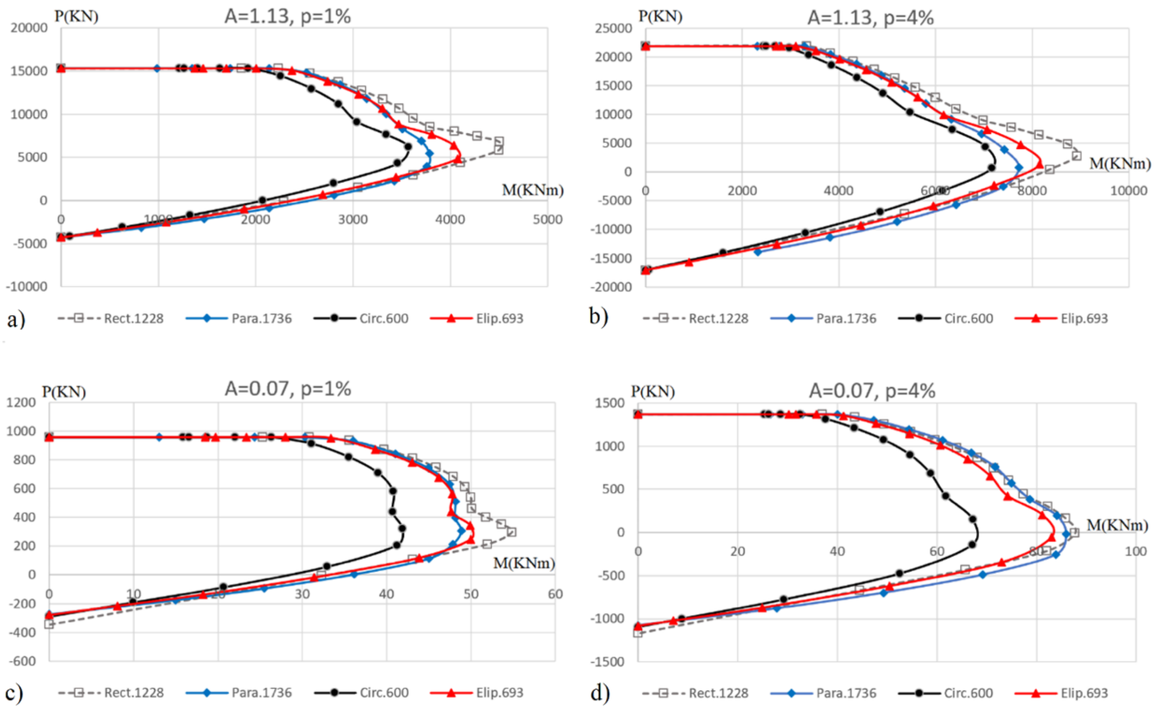

Figure 2. Examples of column interaction diagrams obtained using the Matlab® code for: (a) A = 1.13 m^2, p = 1%; (b) A = 1.13 m^2, p = 4%; (c) A = 0.07 m^2, p = 1%; (d) A = 0.07 m^2, p = 1%.

To make clear the trend of the relative capacity, maximum value of bending moment value of each section has been divided by the respective value of the comparable rectangular section. The same operation has been applied to the axial load corresponding to the maximum value of bending moment. Table 2 shows the relative capacity of sections with respect to rectangular sections.

5. Discussion

Using the results summarized in Table 1, it is possible to identify the relative stiffness of each section with respect to the rectangular section. If values of the moment of inertia of each axis of the section are divided by the respective values of a comparable rectangular section, a relative stiffnes number Irel can be obtained. From these calculations, it is evident that circle, ellipse, and parallelogram forms have relative stiffness of 72%, 95% and 100%. Cicular sections resulted to be the least rigid sections while parallelogram shaped sections showed the same stiffnes than rectangular sections. Interestingly, the elliptical sections were seen to have a high stiffness comparable to the rectangular sections. According to this, elliptical, parallelogram-shaped, and rectangular sections could offer comparable advantages when using them in columns of earthquake resistant systems.

From an analysis of Table 2, it can be concluded that elliptical sections are able to support around 91% of the maximum bending moment supported by comparable rectangular sections. On the other hand, parallelogram shaped sections and circular sections can develop up to 87% and 78% of the maximum bending moment supported by comparable rectangular sections. As for the comparations related to the possible axial load capacities for such maximum bending moments, it has been observed that circular, parallelogram shaped and elliptical sections are able to develop about 87%, 73% and 65% of the values associated to comparable rectangular sections.

To validate the model, experimental tests results taken from literature were used. For example, when comparing the bending and axial force capacity forecasted by our model with those reported by Hadi, *et al.* [2] for columns with no polimeric jacket shaped, using a circular section of 150 mm of diameter with a steel ratio p = 2.7%, we obtained bending moment values of M = 8.7 kNm and M = 11.0 kNm for eccentricities with repect to the diameter of e = 10% and e = 17% repectively, values that show a gigh coincidence with those reported previously [2].

On the other side, to verify the simulation capability of our model, we converted the data of the experiment developed by Jamaluddin, *et al.* [10], who used elliptical concrete filled tube columns of 150x75x4 mm and 200x100x5 mm built concrete of f´c = 47 to 106 MPa. To simulate the presence of steel in the contour, we applied a steel ratio p = 15% as one of the input data. After running the model, we obtained axial force values ranging from 503 KN to 1161 KN which fit quite well with those reported in the cited study [10]. Furthermore, after using the simplified method proposed by Mustafa and Houshiar in 2018, the values of bending moment forecasted by our model resulted similar to those computed from the mentioned method [19]. Variation of results were about 6% to 10% when comparing those of our model with those of the simplified method.

Table 2. Relative capacity of sectios with respect to rectangular sections.

Steel ratio	1%		2%		3%		4%	
Section	M	P	M	P	M	P	M	P
Parallelogram1736	84%	80%	85%	60%	86%	49%	87%	29%
Parallelogram1302	84%	93%	85%	60%	86%	48%	86%	27%
Parallelogram868	83%	92%	84%	59%	85%	46%	86%	22%
Parallelogram434	89%	103%	92%	138%	95%	64%	98%	206%
Circle600	79%	91%	79%	116%	80%	49%	80%	24%
Circle450	78%	105%	78%	63%	79%	46%	80%	17%
Circle300	76%	103%	77%	59%	77%	37%	78%	N.A.
Circle150	76%	108%	76%	136%	77%	222%	77%	N.A.
Ellipse693	91%	71%	91%	76%	91%	66%	91%	47%
Ellipse520	91%	83%	91%	76%	91%	64%	91%	44%
Ellipse346	89%	81%	90%	73%	90%	59%	90%	32%
Ellipse173	91%	83%	93%	75%	94%	49%	95%	698%
Average	84%	91%	85%	82%	86%	67%	87%	115%
Coef. of variation	7%	13%	7%	36%	8%	75%	8%	186%

6. Conclusions

In this work physical–mathematical model has been elaborated and solved to determine the structural efficiency in terms of relative strength and stiffness of sections shaped as ellipses and parallelograms when used in reinforced concrete columns. To do so, section area val1ues from 0.071 m^2 to 1.131 m^2 and steel reinforcement ratios of 1%, 2%, 3%, and 4% have been used.

Sections configurated using circle, ellipse, and parallelogram resulted to have relative stiffness of 72%, 95% and 100% with respect to comparable rectangular sections. Cicular sections resulted to be the least rigid sections while parallelogram shaped sections showed the same stiffnes than rectangular sections. Elliptical sections resulted to have a high stiffness comparable to the rectangular sections. As a conclusion it can be said that elliptical, parallelogram shaped, and rectangular sections are able to offer comparable advantages when considering them in columns which belong to earthquake resistant systems. From the resulting column interaction diagrams, it is evident that the maximum bending moment capacity is only possible for low values of axial load capacity.

Elliptical sections resulted to be able to support around 91% of the maximum bending moment supported by comparable rectangular sections. This proportion diminishes to 87% and 78% in the case of parallelogram shaped sections and circular sections, respectively. The axial load capacities associated to the maximum bending moments resulted to be of about 87%, 73%, and 65% of the values of comparable rectangular sections for circular, parallelogram shaped and elliptical, respectively.

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